Lecture 24

Conditional Distribution: Continuous Case:

Suppose that X and Y are jointly continuous with density $f(x,y)$. Then the conditional density of X given $Y=y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}. \quad \Big(\text{given that } f_Y(y) > 0.$$

For any set A, we have

$$P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx,$$

and so we may define the <u>cumulative conditional distribution</u> as

$$F_{X|Y}(a) = P(X \leq a \mid Y=y)$$
$$= \int_{-\infty}^{a} f_{X|Y}(x|y)\,dx$$

If X and Y are independent, then

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x).$$

Ex: Suppose $f(x,y) = \begin{cases} \frac{e^{-x/y} e^{-y}}{y} & 0 < x, y < \infty \\ 0 & o/w. \end{cases}$

Find $P(X > 1 \mid Y = y)$.

$$P(X > 1 \mid Y = y) = \int_1^\infty f_{X|Y}(x|y)\,dx$$

$$f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)} = \frac{\left(e^{-x/y} \cancel{e^{-y}}\right)/\cancel{y}}{\int_0^\infty \frac{e^{-x/y}\cancel{e^{-y}}}{\cancel{y}}\,dx}$$

$$= e^{-x/y} \Big/ \int_0^\infty e^{-x/y}\,dx$$

$$= \frac{e^{-x/y}}{y}$$

So $P(X > 1 \mid Y = y) = \int_1^\infty \frac{e^{-x/y}}{y}\,dx = -e^{-x/y}\Big|_1^\infty = e^{-1/y}$.

Conditional Probabilities where the distributions are neither jointly continuous

- Suppose that X is a CRV with density $f(x)$
- let N be a DRV.

Then:

$$\frac{P(x < X < x + dx \mid N = n)}{dx}$$

$$= \frac{P(N = n, x < X < x + dx)}{P(N = n)\,dx}$$

$$= \frac{P(N=n \mid x < X < x+dx)}{P(N=n)} \underbrace{\frac{P(x<X<x+dx)}{dx}}_{dx\to 0} \longrightarrow f(x)$$

Letting $dx \to 0$, we get

$$\lim_{dx\to 0} \frac{P(x<X<x+dx \mid N=n)}{dx} = \frac{P(N=n \mid X=x)}{P(N=n)} f(x)$$

and so $$f_{X|n}(x|n) = \frac{P(N=n \mid X=x)}{P(N=n)} f(x).$$

<u>Ex:</u> Consider $n+m$ Bernoulli trials all with the same probability of success. Let X be the random variable on $(0,1)$, and suppose that the trials are successful with probability X. Let $N=$ # of successes among the trials. What is the conditional probability density of X given that $N=n$?

Given that $X=x$, $N = \text{Bin}(n+m, x)$, hence

$$f_{X|N}(x|n) = \frac{P(N=n \mid X=x)\, f_X(x)}{P(N=n)}$$

$$= \begin{cases} \dfrac{\binom{m+n}{n} x^n (1-x)^m}{P(N=n)} & x \in (0,1) \\ 0 & x \notin (0,1). \end{cases}$$

$$= C\, x^n (1-x)^m \quad , x \in (0,1)$$

Since $\int_{-\infty}^{\infty} f_{X|N}(x|n)\,dx = 1$, $C = B(n+1, m+1)$.

So $f_{X|N}(x|n)$ is the distribution function of a β random variable.

Sometimes we want to know the conditional distribution of a random variable X given that X is in some set A.

<u>Discrete case:</u>

$$P(X=x \mid X\in A) = \frac{P(\cancel{X}=x, X\in A)}{P(X\in A)} = \begin{cases} \frac{P(X=x)}{P(X\in A)} & \text{if } x\in A \\ 0 & \text{if } x\notin A \end{cases}$$

<u>Continuous case:</u>

$$f_{X|X\in A}(x) = \frac{f(x)}{P(X\in A)} = \begin{cases} \frac{f(x)}{\int_A f(y)\,dy} & x\in A. \\ 0 & \text{o/w} \end{cases}$$

<u>Ex:</u> A Pareto random variable with positive params (a, λ) has distribution

$$F(x) = 1 - a^{\lambda} x^{-\lambda} \quad x > a.$$

and density $f(x) = \lambda a^{\lambda} x^{-\lambda-1}$, $x > a$.

Suppose $x_0 > a$.

Then $$f_{X|X>x_0}(x) = \frac{f(x)}{P(X>x_0)} = \frac{f(x)}{1-F(x_0)}$$

$$= \frac{\lambda a^{\lambda} x^{-\lambda-1}}{\cancel{1} - (\cancel{1} - a^{\lambda} x_0^{-\lambda})} = \frac{\lambda \cancel{a^{\lambda}} x^{-\lambda-1}}{\cancel{a^{\lambda}} x_0^{-\lambda}}$$

$$= \lambda x_0^{\lambda} x^{-\lambda-1}$$

and so the distribution is again Pareto with new parameters $(x_0 \lambda$.